The irregularity of cyclic multiple planes after Zariski

Daniel Naie

Mathematics Subject Classification (2000): 14E20, 14E22, 14B05

Abstract

A formula for the irregularity of a cyclic multiple plane associated to a branch curve that has arbitrary singularities and is transverse to the line at infinity is established. The irregularity is expressed as a sum of superabundances of linear systems associated to some multiplier ideals of the branch curve and the proof rests on the theory of standard cyclic coverings. Explicit computations of multiplier ideals are performed and some applications are presented.

Introduction

Let $f(x,y)=0$ be an affine equation of a curve $B \subset \mathbb{P}^2$ and H_∞ be the line at infinity. The projective surface $S_0 \subset \mathbb{P}^3$ defined by the affine equation $z^n = f(x,y)$ is called the n-cyclic multiple plane associated to B and H_∞. In [20], Zariski obtains his famous result for the irregularity of certain n-cyclic multiple planes.

Zariski's Theorem. *Let B be an irreducible curve of degree b, transverse to the line at infinity H_∞ and with only nodes and cusps as singularities. Let $S_0 \subset \mathbb{P}^3$ be the n-cyclic multiple plane associated to B and H_∞, and let S be a desingularization of S_0. The surface S is irregular if and only if n and b are both divisible by 6 and the linear system of curves of degree $5b/6-3$ passing through the cusps of B is superabundant. In this case,*

$$q(S) = h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}}(-3+\frac{5b}{6})),$$

where $\mathcal{Z}$ is the support of the set of cusps.

The aim of this paper is to present a generalization of Zariski's Theorem to a branch curve that has arbitrary singularities and is transverse to the line at infinity bringing to the fore the theory of cyclic coverings as developed in [18]. The irregularity will be expressed as a sum of superabundances of linear systems associated to some multiplier ideals of the branch curve B. We refer to [4] for the notion of multiplier ideal. To state the main result in Section 2, we recall here that if the rational c varies from a very small positive value to 1, then one can attach a collection of multiplier ideals $\mathcal{J}(cB)$ that starts at $\mathcal{O}_{\mathbb{P}^2}$, diminishes exactly when c equals a jumping number—they represent an increasing discrete sequence of rationals—and finally ends at $\mathcal{I}_B$.

Theorem (2.1). *Let B be a plane curve of degree b and let H_∞ be a line transverse to B. Let S be a desingularization of the n-cyclic multiple plane associated to B and H_∞. If $J(B,n)$ is the subset of subunitary jumping numbers of B that live in $\frac{1}{\gcd(b,n)}\mathbb{Z}$, then*

$$q(S) = \sum_{\xi \in J(B,n)} h^1(\mathbb{P}^2, \mathcal{I}_{Z(\xi B)}(-3+\xi b)),$$

where $Z(\xi B)$ is the subscheme defined by the multiplier ideal $\mathcal{J}(\xi B)$.

In case the singularities of B are locally given by $x^p = y^q$ such as equations, explicit computations of the jumping numbers and of the multiplier ideals will enable us to apply the above theorem to various examples in Section 4. An example in Remark 4.4 shows that the irregularity may jump in case the position of H_∞ with respect to B becomes special.

Generalizations of Zariski's Theorem are discussed in several papers and the proofs are based on different points of view. First, Zariski's original argument divides naturally into three parts. He describes the canonical system of S in terms of the conditions imposed by the singularities of S_0 that correspond to the cusps. Then he establishes the formula

$$q(S) = \sum_{k=n-\lfloor n/6 \rfloor}^{n-1} h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}}(-3+\left\lceil \frac{kb}{n} \right\rceil)), \tag{1}$$

where $\mathcal{Z}$ denotes the support of the set of cusps. To finish, he invokes the topological result proved in [21]: *If n is the power of a prime and B is irreducible, then the n-cyclic multiple plane is regular.* The theorem follows from the examination of the terms that vanish in the previous sum when the degree of the cyclic multiple plane covers an unbounded sequence of powers of primes.

Second, in [5], Esnault establishes a formula, similar to (1), for the irregularity of the b-cyclic multiple plane S_0, where b is the degree of the branch curve B that possesses arbitrary isolated singularities. She uses the techniques of logarithmic differential complexes, the existence of a mixed Hodge structure on the complex cohomology of the associated Milnor fibre—the complement of S_0 with respect to the plane that contains B—and Kawamata-Viehweg Vanishing Theorem. In [1], Artal-Bartolo interprets Esnault's formula for irregularity and applies it to produce two new Zariski pairs. Two plane curves $B_1, B_2 \subset \mathbb{P}^2$ are called a Zariski pair if they have the same degree and homeomorphic tubular neighbourhood in $\mathbb{P}^2$, but the pairs $(\mathbb{P}^2, B_1)$ and $(\mathbb{P}^2, B_2)$ are not homeomorphic. Zariski was the first to discover that there are two types of plane sextics with six cuspidal singularities: there are the ones where the cusps lie on a conic, and the ones where the cusps don't lie on a plane conic. In [19], Vaquié gives a formula for the irregularity of a cyclic covering of degree n of a nonsingular algebraic surface Y ramified along a reduced curve B of degree b with respect to some projective embedding and a nonsingular hyperplane section H that intersects B transversally. His formula is stated in terms of superabundances of the set of singularities of B and the proof also uses the techniques of logarithmic differential complexes. The superabundances involved are given by ideal sheaves that coincide in fact to the multiplier ideals. Vaquié's paper is one among several to introduce the notion of multiplier ideals implicitly and we refer to [4] for this issue.

Third, in [11], Libgober applies methods from knot theory to study the n-multiple plane S_0. His results are expressed in terms of Alexander polynomials and extend Zariski's Theorem to irreducible curves B with arbitrary singularities and to lines H_∞ with arbitrary position

with respect to B. Later on, in [12, 13, 14], he deals with the case of reducible curves B having transverse intersection with the line at infinity and the irregularity of the multiple plane is expressed using quasiadjunction ideals. The technique is based on mixed Hodge theory, and the result is a particular case in a vaster study, pursued in the above mentioned papers, where the homotopy groups of the complements of various divisors in smooth projective varieties are explored. These groups are related to the Hodge numbers of cyclic or more generally abelian coverings ramified along the considered divisors, as well as to the position of their singularities. We refer the reader to [16] for more ample details and references and to [15] for the relation between the quasiadjunction ideals and the multiplier ideals.

Our argument will follow Zariski's ideas. The multiple plane is transformed into a standard cyclic covering of the plane through a sequence of blowing-ups of $\mathbb{P}^3$. Then an analog of the formula (1) is obtained thanks to the theory of cyclic coverings:

$$q(S) = \sum_{k=1}^{n-1} h^1(\mathbb{P}^2, \mathcal{I}_{Z(\frac{k}{n}B)}(-3 + \left\lceil \frac{kb}{n} \right\rceil)).$$

Finally Theorem 2.1 is established using Kawamata-Viehweg-Nadel Vanishing Theorem.

Remark. The above formula coincides with Vaquié's in [19] when the latter is interpreted for a plane curve B and a line H transverse to it. At the same time, Vaquié's formula in its general form might be obtained by the argument we make use of in establishing Theorem 2.1 if Vaquié's general setting were to be considered.

The paper is organized as follows. In Section 1 the theory of cyclic coverings and some facts about multiplier ideals are recalled. Next, in Section 2 it is shown how through a sequence of blowing-ups a cyclic multiple plane is transformed into a standard cyclic covering of the plane and Theorem 2.1 is proved. Explicit computations of the jumping numbers and multiplier ideals are performed in Section 3, using the theory of clusters. Finally, in Section 4 some applications are presented.

Notation and conventions. All varieties are assumed to be defined over $\mathbb{C}$. Standard symbols and notation in algebraic geometry will be freely used. Moreover, if D is a divisor on the variety X, we shall often write $H^i(X, D)$ and $h^i(X, D)$ instead of $H^i(X, \mathcal{O}_X(D))$ and $h^i(X, \mathcal{O}_X(D))$ respectively. If $\mathcal{L}$ is an invertible sheaf on X, then we shall regularly denote by L a divisor such that $\mathcal{L} \simeq \mathcal{O}_X(L)$.

Acknowledgements. I started this paper during a one week stay at the University of Pisa in the spring of 2004. I would like to thank Rita Pardini for her hospitality and for the friendly talks we had.

The paper owes Mihnea Popa its present form. I would like to record my debt to his reading of a preliminary version in the autumn of 2005 and to his encouragements to generalize the results I obtained at that time.

Finally, I would like to thank my colleagues Laurent Evain and Jean-Philippe Monnier for the conversations they put up with throughout this period.

1 Preliminaries

We shall summarize, in a form convenient for further use, some properties of cyclic coverings and of multiplier ideals.

1.1 Cyclic coverings

Let X be a variety and let G be the finite abelian group of order n. If G acts faithfully on X, then the quotient $Y = X/G$ exists and X is called an abelian covering of Y with group G. The map $\pi : X \to Y$ is a finite morphism, $\pi_*\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_Y$-algebras, and $X \simeq \mathbf{Spec}\,(\pi_*\mathcal{O}_X)$.

If X is normal and Y is smooth, then π is flat which is equivalent to $\pi_*\mathcal{O}_X$ locally free. The action of G on $\pi_*\mathcal{O}_X$ decomposes it into the direct sum of eigensheaves associated to the characters $\chi \in \widehat{G}$,

$$\pi_*\mathcal{O}_X = \bigoplus_{\chi\in\widehat{G}} \mathcal{L}_\chi^{-1}.$$

The action of G on $\mathcal{L}_\chi$ is the multiplication by χ and $\mathcal{L}_1 = \mathcal{O}_Y$.

To understand the ring structure of $\pi_*\mathcal{O}_X$ we suppose that every component D of the ramification locus is 1-codimensional. Such a component is associated to its stabilizer subgroup $H \subset G$ and to a character $\psi \in \widehat{H}$ that generates $\widehat{H}$: ψ corresponds to the induced representation of H on the cotangent space to X at D. Dualizing the inclusion $H \subset G$, such a couple (H, ψ) is equivalent to a group epimorphism $f : \widehat{G} \to \mathbb{Z}/m_f$, where $m_f = |H|$; for any $\chi \in \widehat{G}$, the induced representation $\chi|_H$ is given by $\psi^{f(\chi)^\bullet}$.

Here and later on, $a^\bullet$ denotes the smallest non-negative integer in the equivalence class of $a \in \mathbb{Z}/m$, and $\mathfrak{F}$ the set of all group epimorphisms from $\widehat{G}$ to different $\mathbb{Z}/m\mathbb{Z}$.

Let B_f be the divisor whose components belong to the branch locus and are exactly those covered by components of the ramification locus associated to the group epimorphism f. The ring structure is given by the following isomorphisms (see [18]): for any $\chi, \chi' \in \widehat{G}$,

$$\mathcal{L}_\chi \otimes \mathcal{L}_{\chi'} \simeq \mathcal{L}_{\chi\chi'} \otimes \bigotimes_{f\in\mathfrak{F}} \mathcal{O}_Y(\varepsilon(f,\chi,\chi')B_f) \tag{2}$$

with $\varepsilon(f,\chi,\chi') = 0$ or 1, depending on whether or not $f(\chi)^\bullet + f(\chi')^\bullet < |\operatorname{Im} \hat{f}|$.

The next proposition is formulated for cyclic groups, since it is in this case that will be used in the sequel. We refer again to [18] for the case of abelian groups.

Proposition 1.1. *Let $\pi : X \to Y$ be a cyclic covering with X normal, Y smooth and every component of the ramification locus 1-codimensional. If χ generates $\widehat{G}$, then for every $k = 1, \ldots, n$,*

$$L_{\chi^k} \equiv kL_\chi - \sum_{f\in\mathfrak{F}} \left\lfloor \frac{kf(\chi)^\bullet}{m_f} \right\rfloor B_f. \tag{3}$$

In particular, for $k = n$ equation (3) becomes

$$nL_\chi \equiv \sum_{f\in\mathfrak{F}} [G : \operatorname{Im} \hat{f}]\, f(\chi)^\bullet B_f. \tag{4}$$

Proof. For the proof we need to define the sequence $(\zeta_k^{m,r})_{k\geq 0}$: for m and r fixed positive integers with $r \leq m$, and for $k \geq 0$, put $\zeta_k^{m,r} = 1$ if $\lfloor kr \rfloor_m^{\bullet} < r$, and $\zeta_k^{m,r} = 0$ otherwise. Obviously this sequence is m-periodic, $\zeta_m^{m,r} = \zeta_0^{m,r} = 1$ and in case $m > r$, $\zeta_1^{m,r} = 0$.

Now, from the hypotheses, χ spans the group of characters. Taking $\chi' = \chi^{j-1}$ in (2) we get

$$L_\chi + L_{\chi^{j-1}} \equiv L_{\chi^j} + \sum_{f\in\mathfrak{F}} \zeta_j^{m_f, f(\chi)^\bullet} B_f,$$

since $\varepsilon(f,\chi,\chi^{j-1}) = 1$, $f(\chi)^\bullet + f(\chi^{j-1})^\bullet \geq m_f$, $(f(\chi) + f(\chi^{j-1}))^\bullet < r$ and $[jf(\chi)^\bullet]_{m_f}^\bullet < r$ are equivalent. Then, summing over j from 1 to k,

$$L_{\chi^k} \equiv kL_\chi - \sum_{j=1}^{k}\sum_{f\in\mathfrak{F}} \zeta_j^{m_f, f(\chi)^\bullet} B_f.$$

But $\sum_{j=1}^k \zeta_j^{n,b}$ represents the number of 1's among the first k terms in the sequence $(\zeta_j^{n,b})_j$, hence $\sum_{j=1}^k \zeta_j^{n,b} = \lfloor kb/n \rfloor$ and (3) follows. Formula (4) is obvious, since $\chi^n = 1$. $\square$

Conversely, to every set of data $\mathcal{L}_\chi$, B_f, with $f \in \mathfrak{F}$, that satisfies (4), using (3), we define the line bundles $\mathcal{L}_{\chi^k}$ and associate in a natural way *the standard cyclic covering* π : $\mathbf{Spec}\,(\oplus_k \mathcal{L}_{\chi^k}^{-1}) \to Y$, unique up to isomorphisms of cyclic coverings. The line bundles $\mathcal{L}_{\chi^k}$ verify equation (2) and, consequently, $\oplus_k \mathcal{L}_{\chi^k}^{-1}$ is endowed with a ring structure.

Now, the standard covering thus obtained may not be normal; in fact it is not normal precisely above the multiple components of the branch locus (see [18] Corollary 3.1).

1.2 The normalization procedure for standard cyclic coverings

Let $f : \widehat{G} \to \mathbb{Z}/m_f$ be a group epimorphism, so $m_f = \mathrm{ord}(\mathrm{Im}\,\hat{f})$, and let $B_f = rC + R$, with C irreducible and not a component of R, and $r \geq 2$. X is not normal along the pull-back of C. The normalization procedure along this multiple component of the branch locus splits into three steps showing how to end up with a new covering $X' \to X \to Y$, with X' normal along the pull-back of C (see [18]). The steps are given by the comparison between the multiplicity r and the order m_f of the stabilizer subgroup.

Step 1. If $B_f = rC + R$ with $r \geq m_f$, then set q and r' by the Euclidean division $r = qm_f + r'$, and construct a new set of building data by putting

$$L'_\chi \equiv L_\chi - f(\chi)^\bullet q\, C, \quad B'_f \equiv r'C + R \quad \text{and} \quad B'_g \equiv B_g \quad \text{if } g \neq f.$$

Step 2. If $B_f = rC + R$ with $r < m_f$ and $(r, m_f) = d > 1$, then the natural composition is considered

$$f' : \widehat{G} \xrightarrow{f} \mathbb{Z}/m_f \longrightarrow \mathbb{Z}/\frac{m_f}{d}.$$

The integers $f(\chi)^\bullet$ and $f'(\chi)^\bullet$ are linked by the relation $f(\chi)^\bullet = qm_f/d + f'(\chi)^\bullet$. Put

$$L'_\chi \equiv L_\chi - q\frac{r}{d}C, \quad B'_f \equiv R, \quad B'_{f'} \equiv B_{f'} + \frac{r}{d}C \quad \text{and} \quad B'_g \equiv B_g \quad \text{if } g \neq f, f'$$

in order to construct a 'less non-normal' covering. Notice that the induced multiplicity and the corresponding subgroup order become relatively prime.

Step 3. If $B_f = rC + R$ with $r < m_f$ and $(r, m_f) = 1$, then the composition

$$f' : \widehat{G} \xrightarrow{f} \mathbb{Z}/m_f \xrightarrow{r\cdot} \mathbb{Z}/m_f$$

is considered. As before, the integers $f(\chi)^\bullet$ and $f'(\chi)^\bullet$ are linked by $r \cdot f(\chi)^\bullet = qm_f + f'(\chi)^\bullet$. Put

$$L'_\chi \equiv L_\chi - q\,C, \quad B'_f \equiv R, \quad B'_{f'} \equiv B_{f'} + C \quad \text{and} \quad B'_g \equiv B_g \quad \text{if } g \neq f, f'$$

to get a new covering X' and finish the normalization procedure along C.

Example 1.2. On $\mathbb{P}^2$ let $\mathcal{L}_\chi = \mathcal{O}(1)$ and $nL_\chi \equiv H_0 + (n-1)H_\infty$, where H_0 and H_∞ are two fixed different lines. Here the only functions $f : \widehat{G} \to \mathbb{Z}/n$ involved in (2) are given by $\chi \mapsto 1$ and by $\chi \mapsto n-1$. In this way, the standard n-cyclic covering $S_0 \to \mathbb{P}^2$ is normal and has a singular point above P, the intersection of H_0 and H_∞. To desingularize it, we consider the blow-up surface $\mathrm{Bl}_P\,\mathbb{P}^2$, with E the exceptional divisor and the induced cyclic covering $S \to \mathrm{Bl}_P\,\mathbb{P}^2$. We have $nL_\chi \equiv H_0 + (n-1)H_\infty + nE$ and the induced covering S is not normal above E. The normalization procedure leads to $S' \to \mathrm{Bl}_P\,\mathbb{P}^2$ defined by $nL'_\chi \equiv H_0 + (n-1)H_\infty$, with $L'_\chi \equiv H - E$. S' is a geometrically ruled surface and the pull-back of E is a rational section with self-intersection $-n$, *i.e.* S' is the Hirzebruch surface $\boldsymbol{F}_n$.

1.3 Multiplier ideals

Let X be a smooth variety, $D \subset X$ be an effective $\mathbb{Q}$-divisor and $\mu : Y \to X$ be an embedded resolution for D. Assume that the support of the $\mathbb{Q}$-divisor $K_{Y|X} - \mu^*D$ is a union of irreducible smooth divisors with normal crossing intersections. Then $\mu_*\mathcal{O}_Y(K_{Y|X} - \lfloor \mu^*D \rfloor)$ is an ideal sheaf $\mathcal{J}(D)$ on X. We will denote by $Z(D)$ the subscheme defined by this ideal. Hence $\mathcal{I}_{Z(D)} = \mathcal{J}(D)$. Showing that $\mathcal{J}(D)$ is independent of the choice of the resolution, see [10], we have:

Definition. The ideal $\mathcal{J}(D) = \mu_*\mathcal{O}_Y(K_{Y|X} - \lfloor \mu^*D \rfloor)$ is called the multiplier ideal of D.

The sheaf computing the multiplier ideal verifies the following vanishing result: for every $i > 0$, $R^i\mu_*\mathcal{O}_Y(K_{Y|X} - \lfloor \mu^*D \rfloor) = 0$. Therefore, applying the Leray spectral sequence, we obtain that for every i

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_{Z(D)}) = H^i(Y, \mathcal{O}_Y(\mu^*K_X + \mu^*L + K_{Y|X} - \lfloor \mu^*D \rfloor)). \tag{5}$$

Moreover,

Kawamata-Viehweg-Nadel Vanishing Theorem. *Let X be a smooth projective variety. If L is a Cartier divisor and D is an effective $\mathbb{Q}$-divisor on X such that $L - D$ is a nef and big $\mathbb{Q}$-divisor, then*

$$h^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_{Z(D)}) = 0$$

for every $i > 0$.

Definition-Lemma. (see [4]) Let $B \subset X$ be an effective divisor and $P \in B$ be a fixed point. Then there is an increasing discrete sequence of rational numbers $\xi_i := \xi(B, P)$,

$$0 = \xi_0 < \xi_1 < \cdots$$

such that

$$\mathcal{J}(\xi B)_P = \mathcal{J}(\xi_i B)_P \quad \text{for every} \quad \xi \in [\xi_i, \xi_{i+1}),$$

and $\mathcal{J}(\xi_{i+1} B)_P \subset \mathcal{J}(\xi_i B)_P$. The rational numbers ξ_i's are called *the jumping numbers* of B at P.

2 The irregularity of cyclic multiple planes

Theorem 2.1. *Let B be a plane curve of degree b and let H_∞ be a line transverse to B. Let S be a desingularization of the projective n-cyclic multiple plane associated to B and H_∞. If $J(B, n)$ is the subset of subunitary jumping numbers of B that live in $\frac{1}{\gcd(b,n)}\mathbb{Z}$, then*

$$q(S) = \sum_{\xi \in J(B,n)} h^1(\mathbb{P}^2, \mathcal{I}_{Z(\xi B)}(-3 + \xi b)),$$

with $Z(\xi B)$ the subscheme defined by the multiplier ideal of ξB.

The proof splits naturally into four parts. First, we show that there is a sequence of blowing-ups of $\mathbb{P}^3$ such that S_1, the strict transform of the multiple plane $S_0 \subset \mathbb{P}^3$, becomes a standard cyclic covering of the plane defined by $nL'_\chi \equiv B + (\beta n - b)H_\infty$, with $\beta = \lceil b/n \rceil$ and $\mathcal{L}'_\chi = \mathcal{O}_{\mathbb{P}^2}(\beta)$. Second we choose a desingularization of B such that its total transform on $\mu : Y \to \mathbb{P}^2$ is a divisor with normal crossing intersections, a log resolution. It induces a standard cyclic covering S_2. We apply the normalization procedure to it and obtain a normalization S of S_0, defined by the line bundle $\mathcal{L}_\chi$ and that has only Hirzebruch-Jung singularities.

$$\begin{tikzcd}
S \arrow[dr] \arrow[ddr, "\pi"'] & & & \\
& S_2 \arrow[r] \arrow[d] & S_1 \arrow[r] \arrow[d] & S_0 \\
& Y \arrow[r, "\mu"] & \mathbb{P}^2 &
\end{tikzcd}$$

Third, we compute the line bundles $\mathcal{L}_{\chi^k}$'s in terms of the pull-back $\mu^*\mathcal{O}_{\mathbb{P}^2}(1)$ and the exceptional configuration on Y and get the irregularity of S as a sum of some h^1's. Finally, the result is obtained by applying the Kawamata-Viehweg-Nadel Vanishing Theorem.

The first step is given by:

Proposition 2.2. *Let S_0 be the n-multiple plane associated to the curve B of degree b and the line H_∞. There exists a sequence of blowing-ups $S_1 \to S_0$ such that S_1 is the standard cyclic covering of the plane determined by*

$$nL'_\chi \equiv B + (\beta n - b)H_\infty,$$

with $\beta = \lceil b/n \rceil$ and $L'_\chi \equiv \beta H$.

Proof. Let $[x_0, \ldots, x_3]$ be a homogeneous system of coordinates in $\mathbb{P}^3$. Let Λ be the plane defined by $x_3 = 0$, and let $B, H_\infty \subset \Lambda$ be defined by $F(x_0, x_1, x_2) = 0$ and $x_0 = 0$, respectively. H_∞ will be called the line at infinity. The projective n-cyclic multiple plane $S_0 \subset \mathbb{P}^3$ is defined by $x_3^n = x_0^{n-b} F(x_0, x_1, x_2)$.

If $\deg B = b \leq n$, then things are easy. The point N of homogeneous coordinates $[0, 0, 0, 1]$ is not on S_0. The complement of the exceptional divisor E in the blow-up of $\mathbb{P}^3$ at N coincides with the total space of the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$. Over any open subset $x_i \neq 0$ of the projective plane Λ, $i = 0, 1, 2$, if $z = x_3/x_i$, then z coincides with the tautological section of $p^*\mathcal{O}_\Lambda(1)$ with $p : \mathrm{Bl}_N \mathbb{P}^3 - E \to \Lambda$. The zero divisor of $p^*F - z$ defines S_0. Hence S_0 is the standard cyclic covering determined by

$$nL'_\chi \equiv B + (n - b)H_\infty,$$

with $L'_\chi \equiv H$.

If $\deg B = b > n$, then the situation is slightly more complicated since now N lies on S_0. Let Ξ be the plane spanned by H_∞ and N. In the open set $x_3 \neq 0$, S_0 is defined by

$$u_0^{b-n} = F(u_0, u_1, u_2),$$

with $u_i = x_i/x_3$, $0 \leq i \leq 2$. First, we blow up the projective space at N, $X_1 = \mathrm{Bl}_N \mathbb{P}^3 \to \mathbb{P}^3$. $E \subset \mathrm{Bl}_N \mathbb{P}^3$ denotes again the exceptional divisor and $L_\infty \subset E$ the line that correspond to Ξ, *i.e.* $L_\infty = \Xi \cap E$. The strict transform S_0 is defined by

$$u_0^{(1)\, b-n} = u_1^{(1)\, n} F(u_0^{(1)}, 1, u_2^{(1)})$$

on the subset $u_0 = u_0^{(1)} u_1^{(1)}$, $u_1 = u_1^{(1)}$, $u_2 = u_1^{(1)} u_2^{(1)}$. Notice that the line $L_\infty : u_0^{(1)} = u_1^{(1)} = 0$ is contained in S_0. What we have to understand is the geometry of S_0 along L_∞.

Second, we see X_1 as $\mathbb{P}(\mathcal{O}_\Lambda \otimes \mathcal{O}_\Lambda(1)) \to \Lambda$ and make an elementary transform of X_1 along L_∞. We blow up X_1 along L_∞ (the trace of the new exceptional divisor on E is denoted by L_∞). Then, we contract the strict transform of Ξ to L_∞. We obtain $X_2 = \mathbb{P}(\mathcal{O}_\Lambda \otimes \mathcal{O}_\Lambda(2)) \to \Lambda$. The new exceptional divisor becomes an $\boldsymbol{F}_1$ through H_∞ and L_∞, and will be denoted by Ξ. On $u_0^{(1)} = u_0^{(2)}$, $u_1^{(1)} = u_0^{(2)} u_1^{(2)}$, $u_2^{(1)} = u_2^{(2)}$, an equation for S_0 is

$$u_0^{(2)\, b-2n} = u_1^{(2)\, n} F(u_0^{(2)}, 1, u_2^{(2)}),$$

with $L_\infty : u_0^{(2)} = u_1^{(2)} = 0$.

After $\beta - 1$ elementary transforms along L_∞, we get $S_0 \subset \mathbb{P}(\mathcal{O}_\Lambda \otimes \mathcal{O}_\Lambda(\beta)) \to \Lambda$ with S_0 locally characterized by

$$u_0^{(\beta)\, b-\beta n} = u_1^{(\beta)\, n} F(u_0^{(\beta)}, 1, u_2^{(\beta)}).$$

E is defined by $u_0^{(\beta)} = 0$ and L_∞ by $u_0^{(\beta)} = u_1^{(\beta)} = 0$. The new Ξ is the Hirzebruch surface $\boldsymbol{F}_\beta$. To finish, we put $z = 1/u_1^{(\beta)}$ and look at $x = u_0^{(\beta)}$ and $y = u_2^{(\beta)}$ as to local coordinates on Λ. Then

$$S_0 : z^n = x^{\beta n - b} F(x, 1, y).$$

The complement of E in X_β seen through $p : X_\beta - E \to \Lambda$, coincides with the total space of $\mathcal{O}_\Lambda(\beta)$. The coordinate z coincides with the tautological section of $p^*\mathcal{O}_\Lambda(\beta)$. We conclude that S_0 is the standard cyclic covering determined by

$$nL'_\chi \equiv B + (\beta n - b)H_\infty,$$

with $L'_\chi \equiv \beta H$. □

For the next step in the proof of Theorem 2.1 we need several preliminary results.

Proposition 2.3. *Let Y be smooth and let $\pi : X \to Y$ be a standard cyclic covering of degree n determined by*

$$nL_\chi \equiv \sum_{f\in\mathfrak{F}} [G : \operatorname{Im}\hat{f}]\, f(\chi)^\bullet B_f.$$

For a fixed $g \in \mathfrak{F}$, the branching divisor B_g is supposed to have a multiple component, say $B_g = rC + R$ with $r > 1$. Let $X' \to Y$ be the standard cyclic covering obtained from X after the normalization procedure has been applied to the multiple component rC. If X' is associated to

$$nL'_\chi \equiv \sum_{f\in\mathfrak{F}} [G : \operatorname{Im}\hat{f}]\, f(\chi)^\bullet B'_f,$$

then for every $k = 1, \ldots, n-1$,

$$L'_{\chi^k} \equiv kL_\chi - \left\lfloor \frac{krg(\chi)^\bullet}{m_g} \right\rfloor C - \left\lfloor \frac{kg(\chi)^\bullet}{m_g} \right\rfloor R - \sum_{f\neq g} \left\lfloor \frac{kf(\chi)^\bullet}{m_f} \right\rfloor B_f.$$

Proof. If $r \geq m_g$, then $r = qm_g + r_1$, with $0 \leq r_1 < r$. The covering data are modified to

$$L^{(1)}_\chi \equiv L_\chi - qg(\chi)^\bullet C, \quad B^{(1)}_g \equiv r_1C + R \quad \text{and} \quad B^{(1)}_f \equiv B_f \text{ for } f \neq g. \tag{6}$$

If $(r_1, m_g) = d > 1$, then the map $g_2 : \widehat{G} \xrightarrow{g} \mathbb{Z}/m_g \to \mathbb{Z}/\frac{m_g}{d}$ is considered. The integer $g(\chi)^\bullet$ satisfies

$$g(\chi)^\bullet = q_1\frac{m_g}{d} + g_2(\chi)^\bullet. \tag{7}$$

The covering data are modified to

$$L^{(2)}_\chi \equiv L^{(1)}_\chi - q_1\frac{r_1}{d}C, \quad B^{(2)}_g \equiv R, \quad B^{(2)}_{g_2} \equiv B^{(1)}_{g_2} + \frac{r_1}{d}C \quad \text{and} \quad B^{(2)}_f \equiv B^{(1)}_f \quad \text{for } f \neq g, g_2. \tag{8}$$

Finally, if the multiplicity of C, r_1/d, is an integer greater than 1 and prime to m_g/d, then the map $g_3 : \widehat{G} \xrightarrow{g_2} \mathbb{Z}/m_g \xrightarrow{r_1/d} \mathbb{Z}/\frac{m_g}{d}$ is considered. We have

$$\frac{r_1}{d}g_2(\chi)^\bullet = q_2\frac{m_g}{d} + g_3(\chi)^\bullet. \tag{9}$$

The covering data are modified to

$$L'_\chi \equiv L^{(2)}_\chi - q_2C, \quad B'_{g_2} \equiv B^{(1)}_{g_2}, \quad B'_{g_3} \equiv B^{(2)}_{g_3} + C \quad \text{and} \quad B'_f \equiv B^{(2)}_f \quad \text{for } f \neq g_2, g_3. \tag{10}$$

Using (6), (8) and (10) we have $L'_\chi \equiv L_\chi - (qg(\chi)^\bullet + q_1r_1/d + q_2)C$ and, since we know that $L'_{\chi^k} \equiv kL'_\chi - \sum \lfloor kf(\chi)^\bullet/m_f \rfloor B'_f$, we also have

$$\begin{aligned}
L'_{\chi^k} &\equiv kL'_\chi - \left\lfloor \frac{kg_2(\chi)^\bullet}{m_g/d} \right\rfloor B_{g_2} - \left\lfloor \frac{kg_3(\chi)^\bullet}{m_g/d} \right\rfloor (C + B_{g_3}) - \left\lfloor \frac{kg(\chi)^\bullet}{m_g} \right\rfloor R - \sum_{f\neq g,g_2,g_3} \left\lfloor \frac{kf(\chi)^\bullet}{m_f} \right\rfloor B_f \\
&\equiv kL_\chi - \left(\left\lfloor \frac{kg_3(\chi)^\bullet}{m_g/d} \right\rfloor + kqg(\chi)^\bullet + kq_1\frac{r_1}{d} + kq_2 \right) C - \left\lfloor \frac{kg(\chi)^\bullet}{m_g} \right\rfloor R - \sum_{f\neq g} \left\lfloor \frac{kf(\chi)^\bullet}{m_f} \right\rfloor B_f.
\end{aligned}$$

Now, from (9) and (7), we get successively

$$\left\lfloor \frac{kg_3(\chi)^\bullet}{m_g/d} \right\rfloor = \left\lfloor \frac{kr_1 g_2(\chi)^\bullet}{m_g} \right\rfloor - kq_2 = \left\lfloor \frac{kr_1 g(\chi)^\bullet}{m_g} \right\rfloor - kq_1 \frac{r_1}{d} - kq_2,$$

and finally, by the Euclidean division of r to m_g,

$$\left\lfloor \frac{kg_3(\chi)^\bullet}{m_g/d} \right\rfloor = \left\lfloor \frac{krg(\chi)^\bullet}{m_g} \right\rfloor - kqg(\chi)^\bullet - kq_1 \frac{r_1}{d} - kq_2,$$

□

PROPOSITION 2.4. *Let X be a normal projective variety and Y be a smooth projective variety. Let $\pi : X \to Y$ be a cyclic covering. If ω_X is a dualizing sheaf for X, then*

$$\pi_* \omega_X = \bigoplus_{\chi \in \widehat{G}} \omega_Y \otimes \mathcal{L}_\chi,$$

the action of G on $\omega_Y \otimes \mathcal{L}_\chi$ being the multiplication by χ^{-1}.

Proof. We recall the following construction from [7], III, Ex.6.10 and Ex.7.2 valid for X and Y be projective schemes and $\pi : X \to Y$ a finite morphism. For any quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$, the sheaf $\mathcal{H}om(\pi_*\mathcal{O}_X, \mathcal{G})$ is a quasi-coherent $\pi_*\mathcal{O}_X$-module. Hence there exists a unique quasi-coherent $\mathcal{O}_X$-module, denoted $\pi^!\mathcal{G}$, such that $\pi_*\pi^!\mathcal{G} = \mathcal{H}om(\pi_*\mathcal{O}_X, \mathcal{G})$. If $\mathcal{F}$ is coherent on X and $\mathcal{G}$ is quasi-coherent on Y, then there is a natural isomorphism $\pi_*\mathcal{H}om(\mathcal{F}, \pi^!\mathcal{G}) \simeq \mathcal{H}om(\pi_*\mathcal{F}, \mathcal{G})$. It yields the natural isomorphism

$$\mathrm{Hom}(\mathcal{F}, \pi^!\mathcal{G}) \xrightarrow{\simeq} \mathrm{Hom}(\pi_*\mathcal{F}, \mathcal{G})$$

since $H^0(X, \mathcal{H}om(\mathcal{F}, \pi^!\mathcal{G})) \simeq H^0(Y, \pi_*\mathcal{H}om(\mathcal{F}, \pi^!\mathcal{G}))$. If ω_Y is the canonical sheaf for Y, then it follows that $\pi^!\omega_Y$ is a dualizing sheaf for X. Hence

$$\pi_*\omega_X = \pi_*\pi^!\omega_Y = \mathcal{H}om(\pi_*\mathcal{O}_X, \omega_Y) = \bigoplus_{\chi \in \widehat{G}} \omega_Y \otimes \mathcal{L}_\chi.$$

□

LEMMA 2.5. *Let $S_1 \to Y$ be a normal standard cyclic covering of surfaces defined by the line bundle $\mathcal{L}$. If S_1 has only rational singularities and $S \to S_1$ denotes a desingularization of S_1, then*

$$q(S) = q(Y) + \sum_{k=1}^{n-1} h^1(Y, \omega_Y \otimes \mathcal{L}_{\chi^k}) - \sum_{k=1}^{n-1} h^2(Y, \omega_Y \otimes \mathcal{L}_{\chi^k}).$$

Proof. Since the singularities are rational, if $S \xrightarrow{\varepsilon} S_1$ is a resolution of the singular points of S_1, then $R^i\varepsilon_*\mathcal{O}_S = 0$, for all $i \geq 1$. From the Leray spectral sequence it follows that $h^i(S, \mathcal{O}_S) = h^i(S_1, \mathcal{O}_{S_1})$ for all i, and hence $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{S_1})$. Then $q(S) = q(S_1) = p_g(S_1) + 1 - \chi(\mathcal{O}_{S_1}) = h^0(Y, \pi_*\omega_{S_1}) + 1 - \chi(\pi_*\mathcal{O}_{S_1})$ and using the formulae for $\pi_*\omega_S$ and $\pi_*\mathcal{O}_S$, we get

$$q(\widetilde{S}) = \sum_{k=0}^{n-1} h^0(Y, \omega_Y \otimes \mathcal{L}_{\chi^k}) + 1 - \sum_{k=0}^{n-1} \chi(\mathcal{L}_{\chi^j}^{-1}).$$

By Serre duality, the required equality follows. □

One more notation is in order. Let P be a singular point of B and let $\mu : Y \to \mathbb{P}^2$ be a desingularization of B at P, with $E_{P,1}, E_{P,2}, \ldots$ be the irreducible components of the fibre $\mu^{-1}(P) \subset Y$. $\boldsymbol{E}_P$ will denote this finite array of irreducible curves, and if $\boldsymbol{c}$ is a finite array of rational numbers $c_1, c_2, \ldots$, then

$$\boldsymbol{c} \cdot \boldsymbol{E}_P = \sum_{\alpha} c_\alpha E_{P,\alpha}. \tag{11}$$

Proof of Theorem 2.1. For any integer n, $S_0 \subset \mathbb{P}^3$, the n-cyclic multiple plane associated to B and H_∞ is considered. B is assumed to be reduced and transverse to H_∞. By Proposition 2.2 there is a convenient sequence of blowing-ups such that S_1, the strict transform of S_0, becomes a standard cyclic covering of the plane defined by $nL'_\chi \equiv B + (\beta n - b)H_\infty$, with $\beta = \lceil b/n \rceil$ and $L'_\chi \equiv \beta H$. We choose a desingularization of B such that its total transform by $\mu : Y \to \mathbb{P}^2$ is a divisor with normal crossing intersections. S_1 induces a standard cyclic covering S_2 defined by

$$nL''_\chi \equiv B + (\beta n - b)H_\infty + \sum_P \boldsymbol{c}_P \cdot \boldsymbol{E}_P.$$

We apply the normalization procedure to S_2 to end up with a normalization S of S_0 that has only Hirzebruch-Jung singularities (see [18], Proposition 3.3). By Proposition 2.3, if S is defined by the line bundle $\mathcal{L}_\chi$, then

$$\begin{aligned} L_{\chi^k} &\equiv kL''_\chi - \left\lfloor \frac{k}{n}(\beta n - b) \right\rfloor H - \sum_P \left\lfloor \frac{k}{n}\boldsymbol{c}_P \right\rfloor \cdot \boldsymbol{E}_P \\ &\equiv \left\lceil \frac{kb}{n} \right\rceil H - \sum_P \left\lfloor \frac{k}{n}\boldsymbol{c}_P \right\rfloor \cdot \boldsymbol{E}_P, \end{aligned} \tag{12}$$

the last equality resulting from $\beta k - \lfloor k(\beta n - b)/n \rfloor = \lceil kb/n \rceil$. Here, $\lfloor k\boldsymbol{c}_P/n \rfloor \cdot \boldsymbol{E}_P$ denotes $\sum_\alpha \lfloor kc_{P,\alpha}/n \rfloor E_{P,\alpha}$. From Lemma 2.5, since $H \cdot (-L_{\chi^k}) = -\lceil kb/n \rceil < 0$, it follows that

$$q(S) = \sum_{k=1}^{n-1} h^1(Y, K_Y + L_{\chi^k}). \tag{13}$$

In order to end the proof we have to take account in the formula above, of the vanishing of certain h^1's and of the equality of the others with certain superabundances of linear systems on the projective plane.

Claim. $H^1(Y, \omega_Y \otimes \mathcal{L}_{\chi^k}) \simeq H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3 + \lceil kb/n \rceil) \otimes \mathcal{I}_{Z(k/nB)})$, with $Z(k/nB)$ the scheme defined by the multiplier ideal of k/nB.

Indeed, by (12) and (5), it follows that

$$\begin{aligned}
H^1(Y,\omega_Y\otimes\mathcal{L}_{\chi^k}) &= H^1(Y,\mu^*\omega_{\mathbb{P}^2}\otimes\mathcal{O}_Y(\left\lceil\frac{kb}{n}\right\rceil H)\otimes\mathcal{O}_Y(K_{Y|\mathbb{P}^2}-\sum_P\left\lfloor\frac{k}{n}\boldsymbol{c}_P\right\rfloor\cdot\boldsymbol{E}_P)) \\
&= H^1(Y,\mu^*\omega_{\mathbb{P}^2}\otimes\mathcal{O}_Y(\left\lceil\frac{kb}{n}\right\rceil H)\otimes\mathcal{O}_Y(K_{Y|\mathbb{P}^2}-\left\lfloor\frac{k}{n}\right\rfloor B-\sum_P\left\lfloor\frac{k}{n}\boldsymbol{c}_P\right\rfloor\cdot\boldsymbol{E}_P)) \\
&= H^1(Y,\mu^*\omega_{\mathbb{P}^2}\otimes\mathcal{O}_Y(\left\lceil\frac{kb}{n}\right\rceil H)\otimes\mathcal{O}_Y(K_{Y|\mathbb{P}^2}-\left\lfloor\mu^*\frac{k}{n}B\right\rfloor)) \\
&\simeq H^1(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(-3+\left\lceil\frac{kb}{n}\right\rceil)\otimes\mathcal{I}_{Z(\frac{k}{n}B)}),
\end{aligned}$$

justifying the claim.

Using (13) and the above claim, the irregularity is given by

$$q(S)=\sum_{k=1}^{n-1}h^1(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(-3+\left\lceil\frac{kb}{n}\right\rceil)\otimes\mathcal{I}_{Z(\frac{k}{n}B)}).$$

If $k/n\notin J(B,n)$, then either k/n is not a jumping number of B, or it is, but kb/n is not an integer. In the former case, if ξ is the biggest jumping number for B smaller than k/n, then, since $\lceil kb/n\rceil-\xi>0$,

$$h^1(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(-3+\left\lceil\frac{kb}{n}\right\rceil)\otimes\mathcal{I}_{Z(\frac{k}{n}B)})=h^1(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(-3+\left\lceil\frac{kb}{n}\right\rceil)\otimes\mathcal{I}_{Z(\xi B)})=0$$

by Kawamata-Viehweg-Nadel Vanishing Theorem. In the latter case, we apply the same argument, now using $\lceil kb/n\rceil-kb/n>0$. The result follows. □

Corollary 2.6. *Under the hypotheses of Theorem 2.1, if furthermore B is supposed to be an irreducible plane curve, then*

$$q(S)=\sum_{\xi\in J'(B,n)}h^1(\mathbb{P}^2,\mathcal{I}_{Z(\xi B)}(-3+\xi b)),$$

with $J'(B,n)$ the subset of $J(B,n)$ that contains those rationals ξ for which the denominator can not be the power of a prime.

Proof. In [21] the following topological result is established: *If q is the power of a prime and B is irreducible and transverse to H_∞, then the q-cyclic multiple plane is regular.* By inspecting the formula for the irregularity given in Theorem 2.1 for q-multiple planes associated to B and H_∞, q a power of a prime such that there exists a jumping number $l/q\in J(B,n)$, we obtain the corollary. □

3 The case of specified singularities

Explicit versions of Theorem 2.1 may be formulated as soon as the multiplier ideals and the jumping numbers can be evaluated. Such an explicit version is obtained, for example,

if the singularities of B are locally characterized by the equation $x^{dp} - y^{dq} = 0$, with p, q and d positive integers and p, q relatively prime. Turning to Definition 3.6 where the cluster $K_{p,q}(\alpha, \beta)$ is introduced and if $Z_{p,q}(\alpha, \beta)$ is the subscheme associated to it, we have:

Corollary 3.1. *Let B be a plane curve of degree b with each of its singular points of type, either A_1, or given locally by the equation $x^{dp} - y^{dq} = 0$. Let H_∞ be a line transverse to B and let S be a desingularization of the n-cyclic multiple plane associated to B and H_∞. Then*

$$q(S) = \sum_{(\alpha,\beta)} h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}_{p,q}(\widetilde{\alpha,\beta})}(-3 + \frac{\alpha p + \beta q}{dpq} b)).$$

The sum ranges over the couples (α, β) such that $\dfrac{\alpha p + \beta q}{dpq} < 1$ and $\dfrac{\alpha p + \beta q}{dpq} \in \dfrac{1}{\gcd(b, n)}\mathbb{Z}$. In addition, the couple of positive integers $(\widetilde{\alpha, \beta})$ is defined by

$$\min_{(\alpha', \beta')} \{\alpha' p + \beta' q \geq (\alpha - 1)p + (\beta - 1)q + 1\},$$

and $\mathcal{Z}_{p,q}(\widetilde{\alpha,\beta}) = \cup_P Z_{p,q}(\widetilde{\alpha,\beta})_P$, with $P \in \operatorname{Sing} C$ not of type A_1.

In this section we mainly want to establish this corollary. We need to control the jumping numbers and the multiplier ideals associated to a curve with this type of singularities. The multiplier ideals and their jumping numbers are known in this case; see for example [3] and [4], or [9] for the case of monomial ideals in general. We like to present a different argument based on Enriques diagrams for the particular case of two unknowns, since it will provide a simple interpretation of the multiplier ideals involved, and also, could provide an algorithm for the generalization to an arbitrary singular point of a curve on a surface.

3.1 Clusters and Enriques diagrams

Let X be a surface and $P \in X$ a smooth point. A point Q is called infinitely near to P if $Q \in X'$, $\mu : X' \to X$ is a composition of blowing-ups and Q lies on the exceptional configuration that maps to P.

Definition. A *cluster* in X, centered at a smooth point P is a finite set of weighted infinitely near points to P, $K = \{P_1^{w_1}, \ldots, P_r^{w_r}\}$, with $P_1 = P$.

Let $\mu : Y \to X$ be the composition of blowing-ups $Y = Y_{r+1} \to Y_r \to \cdots \to Y_1 = X$, with $Y_{\alpha+1} = \operatorname{Bl}_{P_\alpha} Y_\alpha$. Since the points infinitely near P are partially ordered—the point Q precedes the point R if and only if R is infinitely near Q—the points of a cluster are partially ordered. In the sequel, if K is a cluster, then all points preceding a point that belongs to K are in K, possibly with weight 0.

Let K be a cluster centered at P. Each point P_α corresponds to an exceptional divisor $E_\alpha \subset Y_{\alpha+1}$. All its strict transforms will also be denoted by E_α and the total transform of each E_α will be denoted by F_α. When needed, the strict transform of E_α on Y_β will be denoted by $E_\alpha^{(\beta)}$, and similarly for the total transform. For example $F_\alpha^{(\alpha+1)} = E_\alpha^{(\alpha+1)}$.

Every cluster K defines a divisor $D_K = \sum w_\alpha F_\alpha$ on Y and an ideal sheaf $\mu_* \mathcal{O}_Y(-D_K)$ on X, hence a subscheme Z_K of X. The lemma below clarifies the comparison between the ideal sheaf $\mathcal{O}_Y(-D_K)$ and the pull-back $\mu^* \mu_* \mathcal{O}_Y(-D_K)$.

Definitions. Let K be a cluster. A point P_β is said to be *proximate* to P_α if P_β lies on E_α, the exceptional divisor corresponding to the blowing-up at P_α, or on one of its strict transforms.

A cluster K is said to satisfy the proximity relations if for every P_α in K,

$$\overline{w}_\alpha = \sum_{P_\beta \text{ proximate to } P_\alpha} w_\beta \leq w_\alpha.$$

LEMMA 3.2 (see also [2], Theorem 4.2). *Let $K = \{P_1^{w_1}, \ldots, P_r^{w_r}\}$ be a cluster that contains a point P_α for which the proximity relation is not satisfied. If $K' = \{P_1^{w'_1}, \ldots, P_r^{w'_r}\}$ is the cluster defined by $w'_\alpha = w_\alpha + 1$, $w'_\beta = w_\beta - 1$ for every β with P_β proximate to P_α, and $w'_\gamma = w_\gamma$ otherwise, then K and K' define the same subscheme in X, i.e. $\mu_*\mathcal{O}_Y(-D_K) = \mu_*\mathcal{O}_Y(-D_{K'})$.*

K' is said to be obtained from K by the unloading procedure. Starting from K, iterated applications of this procedure lead to a cluster $\widetilde{K}$ that satisfies the proximity relations and defines the same subscheme in X. $\widetilde{K}$ is called the unloaded cluster. Notice that

$$\mu^*\mu_*\mathcal{O}_Y(-D_K) \simeq \mu^*\mu_*\mathcal{O}_Y(-D_{\widetilde{K}}) \simeq \mathcal{O}_Y(-D_{\widetilde{K}}).$$

Remark. If $w_r < 0$, then the proximity relation is not satisfied at P_r since $\overline{w}_r = 0$. When the unloading procedure of Lemma 3.2 is applied to a cluster with non-negative weights, it may happen that a weight becomes negative, or more precisely, becomes -1. But it is to be noticed that the negative weight is eventually rubbed out by the next applications of the procedure, and that the unloaded cluster has only non-negative weights. Moreover, the unloaded cluster associated to a cluster with non-positive weights is the empty cluster, the one with all its weights equal to 0.

Definition. A *gridded tree* is a couple (T, g), where $T = T(\mathfrak{V}, \mathfrak{A})$ is an oriented tree with $\mathfrak{V}$ the set of vertices and $\mathfrak{A}$ the set of arcs, and g is a map

$$g : \mathfrak{A} \to \{\text{slant}, \text{horizontal}, \text{vertical}\}.$$

Definition. Let T be a gridded tree. A horizontally (vertically) *L-shape branch* of T is an ordered chain of arcs, such that each begins where the previous ends, and such that all are horizontal (vertical), but the first.

Notice that an L-shape branch is completely determined by the subset of incident vertices of it. Moreover, an arc is an L-shape branch, regardless its value through g.

Definition. Let T be a gridded tree. A *segment* is a maximal chain of arcs of the same type through g, arcs that are also maximal L-shape branches.

EXAMPLE 3.3. Let $p < q$ be relatively prime positive integers. $T_{p,q}$ will denote the gridded tree associated to the Euclidean algorithm. If $r_0 = a_1 r_1 + r_2, \ldots, r_{m-2} = a_{m-1} r_{m-1} + r_m$ and $r_{m-1} = a_m r_m$, with $r_0 = q$ and $r_1 = p$, then $T_{p,q}$ has d segments containing $a_1, \ldots, a_{m-1}$ and respectively a_m vertices each. The first segment is slanted and the others are alternatively, either horizontal or vertical, starting with a horizontal one.

Definition. An *Enriques diagram* is an weighted gridded tree.

Clusters and Enriques diagrams carry the same information as the lemma below asserts, and it will often be convenient to argue using diagrams.

Definition. A point of a cluster is said to be *free* if it is proximate to exactly one point of the cluster. A point is said to be a *satellite* if it is proximate to exactly two points of the cluster.

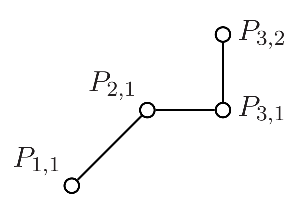
LEMMA 3.4 (see [6]). *There exists an unique map from the set of clusters in X centered at a smooth point P to the set of Enriques diagrams such that:*

1. *for every cluster $K = \{P_1^{w_1}, \dots, P_r^{w_r}\}$ the set of vertices of the image tree is $\mathfrak{V} = \{P_1, \dots, P_r\}$ with the weights given by the integers $w_1, w_2, \dots, w_r$;*
2. *at every point ends at most one arc;*
3. *a point P_α is satellite if and only if there is either a horizontal or a vertical arc that ends at the vertex P_α;*
4. *if there is an arc that begins at the vertex P_α and ends at the vertex P_β then $P_\beta \in E_\alpha^{(\beta)}$, and the converse is true if P_β is free;*
5. *P_β is proximate to P_α if and only if there is an L-shape branch that starts at P_α and ends at P_β;*
6. *the strict transforms E_α and E_β intersect on Y if and only if the Enriques diagram contains a maximal L-shape branch that has P_α and P_β as its extremities;*
7. *an arc that begins at a vertex of a free point and ends at a vertex of a satellite point is horizontal.*

3.2 The minimal unloaded clusters associated to a $T_{p,q}$ tree

Let $p < q$ be relatively prime positive integers. All clusters treated in this subsection will be associated to the gridded tree $T_{p,q}$ introduced in Example 3.3. The intent is to look for a characterization of the minimal unloaded clusters modeled on $T_{p,q}$. We refer to Lemma 3.7 for the result.

Depending on the context its vertices will be denoted either by P_α *i.e.* using one subscript $1 \le \alpha \le r = a_1 + \dots + a_m$, or by $P_{k,i}$, *i.e.* using two subscripts $1 \le k \le d$, $1 \le i \le a_k$. $T_{3,5}$ is represented below in the latter notation.



Let K be a cluster. We define the proximity matrix of K by $\Pi = ||p_{\alpha\beta}||$, where the elements of the diagonal equal 1 and, for every $\alpha \ne \beta$, the element $p_{\alpha\beta}$ equals -1 if P_β is proximate to P_α and 0 if not. Notice that along the α column of Π, the non-zero elements not on the diagonal correspond to the points to which P_α is a satellite.

The proximity matrix is the decomposition matrix of the strict transforms E_α's in terms of the total transforms F_α's. Hence if $K = \{P_1^{w_1}, \ldots, P_r^{w_r}\}$, then on Y,

$$D_K = \sum_\alpha w_\alpha F_\alpha = \sum_\alpha c_\alpha E_\alpha,$$

and $\boldsymbol{c} = \boldsymbol{w}\,\Pi^{-1}$, where $\boldsymbol{w} = (w_1, \ldots, w_r)$ and similarly $\boldsymbol{c} = (c_1, \ldots, c_r)$. The formula

$$E_\alpha = F_\alpha - \sum_{P_\beta \text{ proximate to } P_\alpha} F_\beta$$

and induction on α tell us that the coefficient of E_r in the decomposition of a total transform corresponding to a point lying on the kth segment in terms of strict transforms equals the remainder r_k introduced in Example 3.3.

Lemma 3.5. *If $K = \{P_1^{w_1}, \ldots, P_r^{w_r}\}$ is an unloaded cluster centered at P, then the coefficient of E_r is of the form $ap + bq$, with a, b non-negative integers.*

Proof. We shall denote the weights on the kth segment of the Enriques diagram for the cluster K by $w_{k,1}, w_{k,2}, \ldots, w_{k,a_k}$ and the coefficient of E_r by c_r. We shall successively transform the cluster, each time considering the last segment that contains non-zero weights, unless this segment is the first or the second one. The transformation is the following: if the last segment with non-zero weights is the segment $k+1$, and if $\overline{w}_{k,a_k} = \sum_i w_{k+1,i}$, then put

1. $w'_{k+1,i} = 0$ for $1 \le i \le a_{k+1}$,
2. $w'_{k,i} = w_{k,i} - \overline{w}_{k,a_k}$ for $1 \le i \le a_k$,
3. $w'_{k-1,1} = w_{k-1,1} + \overline{w}_{k,a_k}$,

and leave the other weights unchanged. It is easy to see that the cluster K' defined by the weights $w'_{k,i}$ is again unloaded and that the coefficient of E_r remains unchanged. Hence the same process can be applied till eventually K is transformed into the cluster K' with non-zero weights only on the first and, at the most, the second segments of the Enriques diagram. K' is unloaded and

$$c_r = c'_r = p\sum_{i=1}^{a_1} w'_{1,i} + r_2 \sum_{j=1}^{a_2} w'_{2,j} = p\sum_{i=1}^{a_1}(w'_{1,i} - \overline{w'}_{1,a_1}) + q\overline{w'}_{1,a_1},$$

where $q = a_1 p + r_2$. $\square$

Besides, since for each couple of non-negative integers a, b there exists an unloaded cluster with $ap + bq$ the coefficient of E_r—for example the cluster with $w_{1,1} = a + b$, $w_{1,i} = b$ for $i \neq 1$, $w_{2,1} = b$ and $w_{k,i} = 0$ otherwise—, the ideal sheaf $\mu_* \mathcal{O}_Y(-(ap + bq)E_r)$ defines the minimal unloaded cluster having $ap+bq$ the coefficient of E_r. It is natural to ask the question whether we can decide if an unloaded cluster is minimal only by inspection of its weights, or equivalently its associated divisor. The answer is yes and is given by the lemma hereafter. It will deal with clusters satisfying the following condition:

($*$) for every ordered chain of maximal L-shape branches determined by the points $P_{\alpha_1}, \dots, P_{\alpha_l}$, *i.e.* each P_{α_j} precedes $P_{\alpha_{j+1}}$ and the jth maximal L-shape branch starts at P_{α_j} and ends at $P_{\alpha_{j+1}}$, then

$$\sum_{j=1}^{l}(w_{\alpha_j} - \overline{w}_{\alpha_j}) < \sum_{j=1}^{l} p_{\alpha_j} + 2 - l.$$

Definition 3.6. If a and b are non negative integers, $K_{p,q}(a,b)$ denotes the minimal unloaded cluster associated to the $T_{p,q}$ tree and whose coefficient of the last strict transform equals $ap + bq$.

LEMMA 3.7. *Let K be an unloaded cluster with $ap + bq$ the coefficient of its last strict transform. K satisfies ($*$) if and only if $K = K_{p,q}(a,b)$.*

Proof. We start by showing that a minimal unloaded cluster $K^{\min}$ always satisfies ($*$). Indeed, if not, there would exist a chain of maximal L-shape branches such that $p_{\alpha_j} \geq w_{\alpha_j} - \overline{w}_{\alpha_j}$ for $1 \leq j \leq l$, and that $\sum_1^l w_{\alpha_j} - \overline{w}_{\alpha_j} \geq \sum_1^l (p_{\alpha_j} - 1) + 2$. Furthermore, since there would be at least two points such that $p_\alpha = w_\alpha - \overline{w}_\alpha$, we may always assume $w_{\alpha_1} - \overline{w}_{\alpha_1} = p_{\alpha_1}$, $w_{\alpha_j} - \overline{w}_{\alpha_j} = p_{\alpha_j} - 1$ for $2 \leq j \leq l-1$, and $w_{\alpha_l} - \overline{w}_{\alpha_l} = p_{\alpha_l}$. Now for this chain, we could apply the inverse of the unloading procedure successively at $P_{\alpha_1}, P_{\alpha_2}, \dots, P_{\alpha_l}$ to end up with an unloaded cluster with the same coefficient for E_r as $K^{\min}$, hence a contradiction.

To end the proof, we assume that $K \neq K^{\min}$, K and $K^{\min}$ having the same coefficient for E_r, and show that K does not satisfy ($*$). Since it is satisfied by $K^{\min}$, we notice that along each segment of $K^{\min}$, there is at most one jump of height 1. We may further assume that $w_{1,1} \geq w_{1,1}^{\min} + 1$. To make up for the apparent increase of c_{m,a_m} ($= c_r$) by at least $r_1 = p$ due to the difference between $w_{1,1}$ and $w_{1,1}^{\min}$, some of the weights along the next segments of K must be smaller than the corresponding weights of $K^{\min}$, but not along the first segment. Looking at the points on the second segment and at the first point of the third segment for this counterbalance problem, we notice that at most one of their weights may not diminish, otherwise ($*$) will not be satisfied somewhere along the following segments. Two possibilities can appear. First, all a_2 weights of the second segment satisfy $w_{2,i} \leq w_{2,i}^{\min} - 1$ and, either there exists an i such that $w_{3,i} \leq w_{3,i}^{\min} - 1$, or $w_{3,i} = w_{3,i}^{\min}$ for all i's. In the former case ($*$) is not verified at $P_{1,a_1}, P_{3,1}, P_{3,2}, \dots, P_{3,i}$, and in the latter the same counterbalance problem must be solved for a difference of r_3 units, starting with the fourth segment. Second, the inequalities $w_{2,i} \leq w_{2,i}^{\min} - 1$ are verified for all but one point of the second segment, and $w_{3,1} \leq w_{3,1}^{\min} - 1$. There is a counterbalance problem left for r_2 units, starting with the third segment. Eventually, the counterbalance problem is pushed on to the last segment and hence ($*$) will not be satisfied there for K. □

3.3 The multiplier ideals and the jumping numbers for (x^{dp}, y^{dq})

Let P be a singular point of $B \subset X$ given locally by $x^{dp} + y^{dq} = 0$, with p, q and d positive integers and $p \leq q$ relatively prime, and let $\mu : Y \to X$ be the minimal log resolution of B at P. The exceptional configuration of μ is given by the gridded tree $T_{p,q}$. As before, we shall denote by E_r the last strict transform.

Lemma 3.8. *The coefficient of E_r in $-K_{Y|X} + \lfloor \mu^*\xi B \rfloor$ equals $\lfloor dpq\xi \rfloor - (p+q-1)$.*

Proof. It is sufficient to determine the coefficient of E_r in μ^*B. By Example 3.3 and the decomposition of the F_α's in terms of the strict transforms, we have that the coefficient of E_r is $a_1 r_1^2 + \cdots + a_m r_m^2 = dpq$. □

Proposition 3.9. *If c_r is the coefficient of E_r in $-K_{Y|X} + \lfloor \mu^*\xi B \rfloor$, then the multiplier ideal $\mathcal{J}(\xi B)$ is given by*

$$\mathcal{J}(\xi B) = \mu_* \mathcal{O}_Y(-c_r E_r),$$

i.e. is the ideal sheaf associated to the minimal cluster that contains $c_r E_r$.

Proof. We shall argue on the cluster associated to the divisor $-K_{Y|X} + \lfloor \mu^*\xi B \rfloor$. To find the multiplier ideal is equivalent to determine the unloaded corresponding cluster. Let the pull-back of B be $\sum_1^r c_\alpha E_\alpha + B = \boldsymbol{c} \cdot \boldsymbol{E} + B$. Then $-K_{Y|X} + \lfloor \mu^*\xi B \rfloor = \sum_1^r w_\alpha F_\alpha = \boldsymbol{w} \cdot \boldsymbol{F}$, with $\boldsymbol{w} = -\boldsymbol{\omega} + \lfloor \xi \boldsymbol{c} \rfloor \cdot \Pi$ and $\boldsymbol{\omega} = (1, \ldots, 1)$.

Let $P_{\alpha_1}, \ldots, P_{\alpha_l}$ be ordered points that determine a chain of maximal L-shape branches. Since

$$\boldsymbol{w} - \overline{\boldsymbol{w}} = \boldsymbol{w}\, {}^t\Pi = \lfloor \xi \boldsymbol{c} \rfloor \, \Pi \, {}^t\Pi - \boldsymbol{\omega}\, {}^t\Pi, \tag{14}$$

where the matrix $-\Pi\,\Pi^t$ is the intersection matrix of the strict transforms E_α's on the surface Y, for every $1 \leq j \leq l$,

$$w_{\alpha_j} - \overline{w}_{\alpha_j} = -\left\lfloor \xi c_{\alpha_{j-1}} \right\rfloor + (p_{\alpha_j} + 1) \left\lfloor \xi c_{\alpha_j} \right\rfloor - \left\lfloor \xi c_{\alpha_{j+1}} \right\rfloor + (p_{\alpha_j} - 1).$$

So $\sum_{j=1}^{l} (w_{\alpha_j} - \overline{w}_{\alpha_j})$ equals

$$-\left\lfloor \xi c_{\alpha_0} \right\rfloor + p_{\alpha_1} \left\lfloor \xi c_{\alpha_1} \right\rfloor + \sum_{j=2}^{l-1} (p_{\alpha_i} - 1) \left\lfloor \xi c_{\alpha_j} \right\rfloor + p_{\alpha_l} \left\lfloor \xi c_{\alpha_l} \right\rfloor - \left\lfloor \xi c_{\alpha_{l+1}} \right\rfloor + \sum_{j=1}^{l} (p_{\alpha_j} - 1),$$

and since $\boldsymbol{c}\,\Pi\,{}^t\Pi = (0, \ldots, 0, d)$, we have

$$-2 < \sum_{j=1}^{l} (w_{\alpha_j} - \overline{w}_{\alpha_j}) < \sum_{j=1}^{l} p_{\alpha_j} + 2 - l. \tag{15}$$

Putting $l = 1$ we observe that if the proximity relation is not satisfied at P_α, then $w_\alpha - \overline{w}_\alpha = -1$. But the unloading procedure of Lemma 3.2 at P_α changes the vector $\boldsymbol{w} - \overline{\boldsymbol{w}}$ into the vector $\boldsymbol{w} - \overline{\boldsymbol{w}} + (\Pi\,{}^t\Pi)_\alpha$. It follows that the unloading procedure does not change the inequalities in (15) for the new cluster. We conclude that the unloaded cluster satisfies $(*)$, hence, by Lemma 3.7, the result. □

Proposition 3.10. *The jumping numbers of B at P are $(ap+bq)/(dpq)$ with a, b positive integers.*

Proof. Let ξ be a jumping number and K' the corresponding unloaded cluster and let, by Lemma 3.5, $c'_r = a'p + b'q$ be its coefficient for E_r with a', b' non-negative integers. By Lemma 3.8 and Proposition 3.9, we have

$$\lfloor dpq\xi \rfloor - (p + q - 1) = ap + bq + 1 \leq a'p + b'q$$

where a and b are non-negative integers, and $a'p + b'q$ is the first integer combination of p and q with this property. So, by the definition of the jumping numbers, $\xi = ((a+1)p + (b+1)q)/(dpq)$. □

Proof of Corollary 3.1. We know that the irregularity is given by

$$\sum_{\xi \in J(B,n)} h^1(\mathbb{P}^2, \mathcal{I}_{Z(\xi B)}(-3+\xi b))$$

with $J(B,n)$ the subset of jumping numbers ξ of B of the form k/n, $0<k<n$, and such that ξb is an integer. By Proposition 3.10, it is sufficient to describe the subscheme associated to the multiplier ideal $\mathcal{J}(\xi B)$ for every $\xi = (\alpha p + \beta q)/(dpq) \in J(B,n)$. By Proposition 3.9, Lemma 3.8 and Lemma 3.7, the subscheme is given by the minimal unloaded cluster whose coefficient for the last strict transform is the first integer combination of p and q not smaller than $(\alpha-1)p + (\beta-1)q + 1$. □

Remark 3.11. Since many of the applications in the next section will be for a curve B with singularities of a given type $A_\bullet$, we interpret Corollary 3.1 for this situation. If $P_1, \dots, P_r$ are the infinitely near points to $P = P_1$ involved in the minimal log resolution of an $A_\bullet$ type singularity at P, we shall denote by $Z_P^{[\alpha]}$ the curvilinear subscheme associated to the unloaded cluster $\{P_1, \dots, P_\alpha\}$, and by $\mathcal{Z}^{[\alpha]} = \cup Z_P^{[\alpha]}$.

i) If the singularities of B are of type A_1 or A_{2r-1}, *i.e.* $p=1$, $q=r$ and $d=2$, then

$$q(S) = \sum_\alpha h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}^{[\alpha]}}(-3 + \frac{(\alpha+r)b}{2r})),$$

α ranging from 1 to $r-1$ such that $\frac{\alpha+r}{2r} \in \frac{1}{\gcd(b,n)}\mathbb{Z}$.

ii) If the singularities of B are of type A_1 or A_{2m}, *i.e.* $p=2$, $q=2m+1$ and $d=1$, then

$$q(S) = \sum_\alpha h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}^{[\alpha]}}(-3 + \frac{\alpha b}{2m+1} + \frac{b}{2})),$$

n and b are even, and α ranges from 1 to m such that $\frac{\alpha}{2m+1} \in \frac{1}{\gcd(b,n)}\mathbb{Z}$.

4 Applications

We shall now apply the results in the previous sections to illustrate how to compute in an uniform way, the irregularity for some examples of cyclic multiple planes.

Zariski's example

The curve B is irreducible, of degree 6 and has six cusps as singularities. In the formula for the irregularity of the 6-cyclic multiple plane in Remark 3.11 ii), since $m=1$, α may only be 1. Hence $q(S) = h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}}(2))$, where $\mathcal{Z}$ is the support of the cusps. So either the cusps lie on a conic and the irregularity is 1, or they do not, and the irregularity is 0. Notice that the result is the same for every n-cyclic multiple plane, provided that 6 divides n.

Artal-Bartolo's first example in [1]

Let $C \subset \mathbb{P}^2$ be a smooth elliptic curve and let P_1, P_2, P_3 be three inflexion points of C, with L_i the tangent lines at P_i to C. Taking $B = C + L_1 + L_2 + L_3$ we construct the multiple cyclic plane with three sheets S_0 associated to B and H_∞. The curve B has three points of type A_5 at the P_i's, hence $n = 3$, $b = 6$ and $r = 3$ in Remark 3.11 i). We have

$$q(S) = h^1(\mathbb{P}^2, \mathcal{I}_{\{P_1,P_2,P_3\}}(1)).$$

So, if the three inflexion points are chosen on a line, then the irregularity is 1. If the points are not aligned, then the irregularity is 0. These two configurations give an example of a Zariski pair.

Artal-Bartolo's second example in [1]

Let P be a fixed point and $K = \{P_1, \ldots, P_9\}$ a cluster centered at P, all its points being free. It represents a curvilinear subscheme $Z = Z_K$. In [1], Artal-Bartolo considers sextics with an A_{17} type singularity at P, with $P_2, \ldots, P_9$ the infinitely near points of the minimal resolution.

1) If P_3 lies on the line L determined by P_1 and P_2 and if K does not impose independent conditions on cubics, then all sextics are reducible. Let B be the union of two smooth cubics from $|\mathcal{I}_Z(3)|$, and let H_∞ be a line transverse to B. If S_0 is the 3-cyclic multiple plane associated to B and H_∞, then by Remark 3.11 i),

$$q(S) = h^1(\mathbb{P}^2, \mathcal{I}_{Z^{[3]}}(1)) = 1.$$

Similarly, if S_0 is the 6-cyclic multiple plane, then

$$q(S) = h^1(\mathbb{P}^2, \mathcal{I}_{Z^{[3]}}(1)) + h^1(\mathbb{P}^2, \mathcal{I}_{Z^{[6]}}(2)) = 2,$$

since there is no irreducible conic through $Z^{[6]}$—*i.e.* through the points $P_1, \ldots, P_6$— but the double line $2L$: if $K' = \{P_1^2, P_2^2, P_3^2\}$, then $Z^{[6]} \subset Z_{K'}$.

More generally, if S_0 is the n-cyclic multiple plane associated to B and H_∞, then by the same argument it follows that $q(S) = 2$ when $n \equiv 0 \mod 6$, $q(S) = 1$ when $n \equiv 3 \mod 6$, and $q(S) = 0$ otherwise.

2) If $P_3 \notin L$ and $P_6 \in \Gamma$, the conic through $P_1, \ldots, P_5$, then there exists an irreducible sextic with an A_{17} type singularity at P, such that the intersection with Γ is supported only at P. If S_0 is the n-cyclic multiple plane associated to B and to a transverse line to it, then

$$q(S) = h^1(\mathbb{P}^2, \mathcal{I}_{Z^{[6]}}(2)) = 1$$

when n is divisible by 6, and $q(S) = 0$ otherwise.

3) If $P_3 \notin L$ and $P_6 \notin \Gamma$, the conic through $P_1, \ldots, P_5$, then, for every reduced sextic B with an A_{17} type singularity at P, if S_0 is the n-cyclic multiple plane associated to B and to a transverse line to it, then $q(S) = 0$.

Remark. In [1] it is shown that in the third case above, two configurations may appear: either $P_1, \ldots, P_9$ do not impose independent conditions on cubics and B is the union of two smooth cubics, or the points impose independent conditions on cubics and B is irreducible. Using these and the two configurations in 1) and 2), two more Zariski couples are thus produced there.

Oka's example in [17]

In [17], when p and q are relatively prime integers, Oka constructs the curve $C_{p,q}$ of degree pq enjoying the following property: $C_{p,q}$ has pq cusp singularities each of which is locally defined by the equation $x^p + y^q = 0$. We shall show that the pq-multiple plane associated to $C_{p,q}$ is irregular, the irregularity being equal to $(p-1)(q-1)/2$.

We start with the particular case $p = 2$, since all ideas of the general computation are already present in this situation. The construction of the branching curve $B = C_{2,2m+1}$ is as follows. Let $C \subset \mathbb{P}^2$ be a curve of degree $2m+1$ and let Γ be a conic transverse to C. If $f = 0$ and $g = 0$ are homogeneous equations for C and Γ respectively, then the curve B is defined by $f^2 + g^{2m+1} = 0$. It is a curve of degree $4m+2$ with $4m+2$ singular points of type A_{2m}. Let S_0 be the $(4m+2)$-cyclic multiple plane associated to B and let S be the normal cyclic covering constructed in Section 2.

Claim. $q(S) = m$.
To see this, we apply Remark 3.11 ii) to obtain $q(S) = \sum_{\alpha=1}^{m} h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}^{[\alpha]}}(2m+2\alpha-2))$, where $\mathcal{Z}^{[\alpha]} = \cup_P Z_P^{[\alpha]}$ and $Z_P^{[\alpha]}$ is the curvilinear subscheme associated to the cluster $\{P_1 = P, P_2, \ldots, P_{m+2}\}$. We shall show that all the terms of the sum equal 1. To do this, we apply the trace-residual exact sequence with respect to Γ, see [8] or Remark 4.1, and obtain

$$0 \longrightarrow \mathcal{I}_{\mathcal{Z}^{[\alpha-1]}}(2m+2\alpha-4) \longrightarrow \mathcal{I}_{\mathcal{Z}^{[\alpha]}}(2m+2\alpha-2) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(4\alpha-6) \longrightarrow 0.$$

Since $C \in |\mathcal{I}_{\mathcal{Z}^{[m+1]}}(2m+1)|$, the map $H^0(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}^{[\alpha]}}(2m+2\alpha-2)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4\alpha-6))$ from the long exact sequence in cohomology is surjective for every $1 \le \alpha \le m$. Hence

$$h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}^{[r]}}(4r-2)) = \cdots = h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}^{[1]}}(2r)) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 1.$$

Remark. The irregularity of the n-cyclic multiple plane associated to B and to a line H_∞ transversal to B, n being an arbitrary positive integer, may be computed by the same argument. Of course, if $2m+1$ is a prime number, then $q(S) = 0$ unless $4m+2$ divides n, see Corollary 2.6. But if $2m+1$ is not a prime number, then irregular cyclic multiple planes exist for other values of n. For example, if $2m+1 = 15$ and $n = 40$, then

$$q(S) = h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}^{[3]}}(18)) + h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}^{[6]}}(24)) = 2.$$

In the general case, if $p < q$, let $B = C_{p,q}$ and C_p and C_q be the smooth curves of degree p and respectively q used in the construction of B. C_p and C_q intersect transversely and if P is an intersection point, then B has a singularity at P given locally by an $x^p + y^q = 0$ type equation.

Claim. $q(S) = (p-1)(q-1)/2$.
By Corollary 3.1,

$$q(S) = \sum_{\substack{\alpha,\beta \ge 1 \\ \alpha p + \beta q < pq}} h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}_{p,q}(\widetilde{\alpha,\beta})}(-3 + \alpha p + \beta q)).$$

The sum consists of $(p-1)(q-1)/2$ terms, and as before, we shall show that each of them equals 1. For an arbitrary couple (α, β), with $\alpha \ge 2$, we first apply the trace-residual exact sequence $\alpha - 1$ times with respect to C_p. We have

$$0 \to \mathcal{I}_{\mathcal{Z}(\widetilde{\alpha-1,\beta})}(-3+(\alpha-1)p+\beta q) \to \mathcal{I}_{\mathcal{Z}(\widetilde{\alpha,\beta})}(-3+\alpha p+\beta q) \xrightarrow{\rho} \mathcal{I}_{\mathrm{Tr}_{C_p}\, \mathcal{Z}(\widetilde{\alpha,\beta})}(-3+\alpha p+\beta q) \to 0.$$

Let w_1 be the weight of $P_1 = P$ in the cluster $K_{p,q}(\widetilde{\alpha,\beta})$. Since the cluster is not greater than $K_{p,q}(0, \lfloor(\alpha-1)p/q\rfloor + \beta)$, it is easy to see that $w_1 \leq \lfloor(\alpha-1)p/q\rfloor + \beta$. Then $\mathcal{Z}(\widetilde{\alpha,\beta}) \subset w_1 C_q$ and together with the identity

$$-3 + \alpha p + \beta q = -3 + p + \left((\alpha-1)p - \left\lfloor \frac{(\alpha-1)p}{q} \right\rfloor q\right) + \left(\left\lfloor \frac{(\alpha-1)p}{q} \right\rfloor + \beta - w_1\right) q + w_1 q$$

imply the surjectivity of $H^0\rho$. We conclude that

$$h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}(\widetilde{\alpha,\beta})}(-3 + \alpha p + \beta q)) = h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}(\widetilde{1,\beta})}(-3 + p + \beta q)) \tag{16}$$

whenever $\alpha \geq 2$. Then, in case $\beta \geq 2$, we apply $\beta - 1$ times the trace-residual exact sequence with respect to C_q starting with the subscheme $\mathcal{Z}(\widetilde{1,\beta})$. As before, we have

$$0 \to \mathcal{I}_{\mathcal{Z}(\widetilde{1,\beta-1})}(-3 + p + (\beta-1)q) \to \mathcal{I}_{\mathcal{Z}(\widetilde{1,\beta})}(-3 + p + \beta q) \xrightarrow{\rho} \mathcal{I}_{\mathrm{Tr}_{C_q} \mathcal{Z}(\widetilde{1,\beta})}(-3 + p + \beta q) \to 0,$$

the surjectivity of $H^0\rho$ being given by the inequality $w < 1 + (\beta-1)q/p$, with w the sum of the weights of the points $P_{1,1}, \ldots, P_{1,a_1}, P_{2,1}$ in $K_{p,q}(\widetilde{1,\beta})$ and the inclusion $\mathcal{Z}(\widetilde{1,\beta}) \subset wC_p$. So

$$h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}(\widetilde{1,\beta})}(-3 + p + \beta q)) = h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}(\widetilde{1,1})}(-3 + p + q)). \tag{17}$$

Finally, since $\mathcal{Z}_{p,q}(\widetilde{1,1}) = \cup_P P$, we apply once more the trace-residual exact sequence of $\mathcal{Z}(\widetilde{1,1})$ with respect to C_p and get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-3 + q) \longrightarrow \mathcal{I}_{\mathcal{Z}(\widetilde{1,1})}(-3 + p + q) \longrightarrow \mathcal{O}_{C_p}(-3 + p) \longrightarrow 0.$$

Since $q > p$, $h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}(\widetilde{1,1})}(-3 + p + q)) = h^1(C_p, \mathcal{O}_{C_p}(-3 + p)) = 1$. Together with (16) and (17) finish the claim.

Remark 4.1 (The trace-residual exact sequence). Let X be a projective variety, D be a Cartier divisor on X and Z be a a closed subscheme of X. The schematic intersection $\mathrm{Tr}_D Z = D \cap Z$ defined by the ideal sheaf $(\mathcal{I}_D + \mathcal{I}_Z)/\mathcal{I}_D$ is called the trace of Z on D. The closed subscheme $\mathrm{Res}_D Z \subset X$ defined by the conductor ideal $(\mathcal{I}_Z : \mathcal{I}_D)$ is called the residual of Z with respect to D. Following [8], the canonical exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathrm{Res}\, Z}(-D) \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{I}_{\mathrm{Tr}\, Z} \longrightarrow 0.$$

is called the trace-residual exact sequence of Z with respect to D.

A specialization of Oka's example when $p = 2$

Keeping the notation from the first part of the previous paragraph, the conic Γ is now the union of two distinct lines that intersect at O and C is a smooth curve of degree $2m + 1$ passing through O and intersecting transversely the lines of Γ at this point. The curve B has $4m$ points of type A_{2m} and one singular point at O of type A_{4m+1}. It can be shown that the irregularity of the $(4m + 2)$-cyclic multiple plane associated to B is again m. We develop the computation for $m = 2$. In this case, B is a curve of degree 10 with 8 points of type A_4

and one point of type A_9. By Theorem 2.1 and using the notation from Remark 3.11, the irregularity is given by

$$h^1(\mathbb{P}^2, \mathcal{I}_{\xi^{[1]}\cup Z_O^{[2]}}(4)) + h^1(\mathbb{P}^2, \mathcal{I}_{\xi^{[2]}\cup Z_O^{[4]}}(6)),$$

where $\xi^{[1]}$ is the support of the points of type A_4 and $\xi^{[2]} = \cup_{P \text{ of type } A_4} Z_P^{[2]}$ is the support plus the tangent directions. Now, 10 points on a conic do not impose independent conditions on quartics, hence the first term is 1. The second term is seen to be equal to the first after applying the trace-residual exact sequence with respect to the two lines of Γ. So the irregularity is 2.

The computations for $m = 1$ lead to a branching curve of degree 6 with 4 cusps and an A_5 singularity at O. The irregularity of a 6-cyclic multiple plane is 1, given by $h^1(\mathbb{P}^2, \mathcal{I}_{\xi^{[1]}\cup Z_O^{[2]}}(2))$. If in addition, the two lines of the degenerate conic Γ are brought together such that the cusps collapse two by two, the branching curve has 3 A_5 singularities. For a 6-multiple plane, $q = 2$, with the contributions of the superabundance of the singularities with respect to the lines and the conics both equal to 1. Necessarily, by Corollary 2.6, the branching curve is reducible; it is Artal-Bartolo's first example.

Line arrangements following [5]

In this example we consider as branch curve a line arrangement $B = \cup_{i=1}^{b} L_i \subset \mathbb{P}^2$ that has only nodes and ordinary triple points as singularities. For an ordinary triple point $2/3$ is the only subunitary jumping number. By Corollary 3.1, if H_∞ is a line transverse to $B = \cup_{i=1}^{b} L_i$, then the normal n-cyclic covering S corresponding to the n-cyclic multiple plane associated to B and H_∞ is irregular if and only if 3 divides both b and n, and $|\mathcal{I}_{\mathcal{Z}}(-3+\frac{2b}{3})|$ is superabundant, in which case

$$q(S) = h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}}(-3 + \frac{2b}{3})).$$

In case S is irregular, it can be shown that the irregularity is bounded by a constant depending on the arrangement B. More precisely, we have

Proposition 4.2. *Let $B = \cup_{i=1}^{b} L_i$, H_∞ and S be as above with b and n divisible by 3. If t_i is the number of triple points lying on the line L_i for each i, then*

$$q(S) \le \min_{i=1}^{b} t_i.$$

For the proof (see [5] for a different argument), we will start with a preliminary lemma.

Lemma 4.3. *If 3 divides both b and n and if one line of the arrangement contains no triple point, then $q(S) = 0$.*

Proof. Let B' be the arrangement of the $b - 1$ lines of B except the one one with no triple point. If S' is the normal n-cyclic covering corresponding to the n-cyclic multiple plane associated to B' and H_∞, then $q(S') = 0$ since 3 does not divide $\deg B'$. Taking $k = 2n/3$ and denoting by $\mathcal{Z}$ the support of the triple points, we obtain

$$0 = h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}}(-3 + \left\lceil \frac{2(b-1)}{3} \right\rceil) = h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}}(-3 + \frac{2b}{3}) = q(S).$$

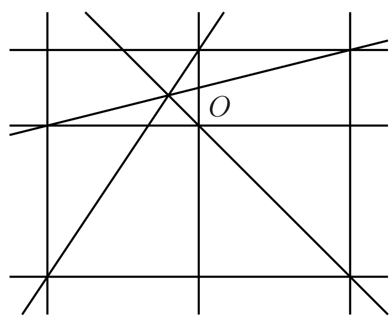
□

Proof of Proposition 4.2. Let us suppose that L_1 is the line containing the minimum number of triple points. If $B' = L'_1 \cup \bigcup_{i \neq 1} L_i$ is a line arrangement with no triple point on L'_1, then by the previous lemma, $h^1(\mathbb{P}^2, \mathcal{I}_{\mathrm{Res}_{L_1} \mathcal{Z}}(-3 + 2b/3)) = 0$. But

$$h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}}(-3 + \frac{2b}{3})) \leq h^1(\mathbb{P}^2, \mathcal{I}_{\mathrm{Res}_{L_1} \mathcal{Z}}(-3 + \frac{2b}{3})) + \mathrm{card}(\mathcal{Z} - \mathrm{Res}_{L_1} \mathcal{Z}) = t_1,$$

hence the result. □

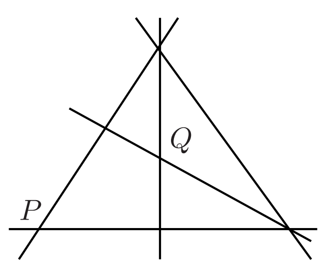
EXAMPLE. Let B be the line arrangement of 9 lines with 9 triple points represented below. In a convenient affine coordinate system (x, y), the triple points that lie in the affine plane are the following: $(0,0), (\pm 2, -2), (-2, 0), (0, s), (2, s)$ and $2s/(s+4)\,(-1, 1)$, with $s \neq -2, 0$ and 2.



It is easy to see that there are two cubics—each the union of three lines—through the 9 triple points, *i.e.* the system of cubics through the points is superabundant. It follows that the irregularity of the n-cyclic multiple plane associated to B and to a line H_∞ transverse to B, is 1 if and only if 3 divides n.

If $s = 2$, then the arrangement specialize to an arrangement with 10 triple points, 4 of them lying on the line $x + y = 0$. But these points lie on a cubic, the union of three of the lines of B, and again $h^1(\mathbb{P}^2, \mathcal{I}_{\mathcal{Z}}(3)) = 1$, hence the irregularity is 1 in this case too.

Remark 4.4. The irregularity depends on the position of the line H_∞ with respect to B. To see this, let B be the line arragement below of 5 lines with 2 triple points.



If H_∞ is transverse to B, then the irregularity of the 6-cyclic multiple plane is 0. But if H_∞ is the line through the double points P and Q then the irregularity jumps to 1.

References

[1] E. Artal-Bartolo, Sur les couples de Zariski. *J. Algebraic Geom. 3* (1994), 223–247.

[2] E. Casa-Alvero, Infinitely near imposed singularities and singularities of polar curves. *Math. Ann. 287* (1990), 429–454.

[3] L. Ein, Multiplier ideals, vanishing theorems and applications. *Algebraic geometry—Santa Cruz 1995*, 203–219.

[4] L. Ein, R. Lazarsfeld, K. E. Smith and D. Varolin, Jumping coefficients of multiplier ideals. *Duke Math. J. 123* no. 3 (2004), 469-506.

[5] H. Esnault, Fibre de Milnor d'un cône sur une courbe algébrique plane. *Invent. Math. 68* (1982), 477–496.

[6] L. Evain, La fonction de Hilbert de la réunion de 4^h gros points gńériques de $\mathbb{P}^2$ de même multiplicité. *J. Algebraic Geom. 8* (1999), 787–796.

[7] R. Hartshorne, *Algebraic Geometry*. Graduate Texts in Mathematics, Springer-Verlag, 1977.

[8] A. Hirschowitz, La méthode d'Horace pour l'interpolation à plusieurs variables. *Manuscripta Math. 50* (1985), 337–388.

[9] J. A. Howald, Multiplier ideals of monomial ideals. *Trans.Amer.Math.Soc. 353* (2001), 2665–2671.

[10] R. Lazarsfeld, *Positivity in algebraic geometry*. A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 2004.

[11] A. Libgober, Alexander polynomial of plane algebraic curves and cyclic multiple planes. *Duke Math. J. 49* (1982), 833–851.

[12] A. Libgober, Homotopy groups of the complements to singular hypersurfaces. *Bull. Amer. Math. Soc. 13* (1985), 49–52.

[13] A. Libgober, Position of singularities of hypersurfaces and the topology of their complements. *J. Math. Sci. 82* (1996), 3194–3210.

[14] A. Libgober, Characteristic varieties of algebraic curves. In *Applications of algebraic geometry to coding theory, physics and computation* (Eilat, 2001), 215–254, NATO Sci. Ser. II Math. Phys. Chem., 36, Kluwer Acad. Publ., Dordrecht, 2001.

[15] A. Libgober, Hodge decomposition of Alexander invariants. *Manuscripta Math. 107* (2002), 251–269.

[16] A. Libgober, Lectures on topology of complements and fundamental groups, arXiv:math.AG/0510049 (2005).

[17] M. Oka, Some plane curves whose complements have nonabelian fundamental groups. *Math. Ann. 218* (1978), 55–65.

[18] R. Pardini, Abelian covers of algebraic varieties. *J. reine angew. Math. 417* (1991), 191–213.

[19] M. Vaquié, Irrégularité des revêtements cycliques des surfaces projectives non singulières. *Amer. J. Math. 114* (1992), no. 6, 1187–1199.

[20] O. Zariski, On the irregularity of cyclic multiple planes. *Ann. of Math. 32* (1931), 485-511.

[21] O. Zariski, On the linear connection index of the algebraic surfaces $z^n = f(x, y)$. *Proceedings Nat. Acad. Sciences 15* (1929), 494-501.

Daniel Naie
Département de Mathématiques
Université d'Angers
F-40045 Angers
France